Lecture 9 - Oct 1

Graphs

Mathematical Induction: Degree Sum
Paths, Cycles, Reachability
(Spanning vs. Connected) Subgraphs

Announcements/Reminders

- First Class (Syllabus) recording & notes posted
- Today's class: notes template posted
- Exercises:
    - + Tutorial Week 1 (2D arrays)
    - + Tutorial Week 2 (2D arrays, Proving Big-O)
    - + Tutorial Week 3 (avg case analysis on doubling strategy)
    - + Tutorial Week 4 (Trinode restructuring after deletions)

Properties: Sum of Degrees for Undirected Graphs

2. $(m+d)$ → # edges in the extended graph

Given a **simple** (non-empty) **undirected** graph $G = (V, E)$ with $|E| = m$:

$$\boxed{\sum_{v \in V} \text{degree}(v) = 2 \cdot m}$$ claim

Strategy of Proof : Perform a M.I. on $|V|$

(1) Base Case : $|V| = 1$

x $\quad |E| = 0.$

degree$(x) = 0$ $\quad \sum_{v \in \{x\}} \text{degree}(v) = \text{degree}(x) = 0 = 2 \cdot \underbrace{|E|}_{0}$

(2) Inductive Hypothesis (I.H.)

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot m$$

where $|V| = \boxed{k > 1}$ more substantial than base cases.

(3)* Make a strictly larger graph with $k+1$ vertices (by adding a new vertex y)

Choose d existing vertices and connect them to y. $d \le k$.

I.H. (excluding y) from y to existing nodes

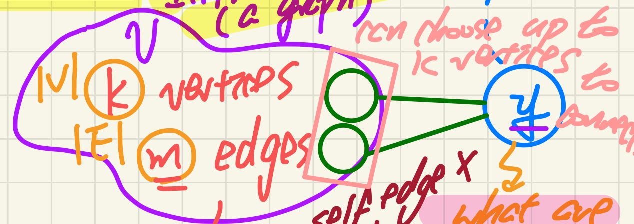
$$\sum_{v \in V \cup \{y\}} \text{degree}(v) = \boxed{2 \cdot m} + d + d^{*}$$

$V \cup \{y\}$: $k+1$ vertices

d^{*}: from existing nodes to y

$\sum_{v \in V} \text{degree}(v) = 2 \cdot m$

I.H. (a graph)



I.H. holds

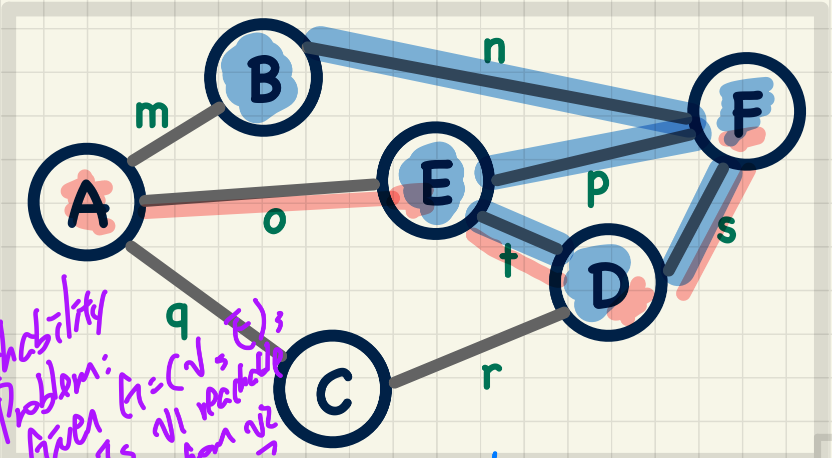
self edge ×

parallel edge ×

what are the possible ways to connect y to the existing graph?

a graph with a cycle ⇒ cyclic
a graph without a cycle ⇒ acyclic

Graph: Paths and Cycles



Path: (F, s, D, t, E, p, F, n, B)

start vertex

end vertex

also a cycle.

Simple Path: (F, s, D, t, E, o, A)

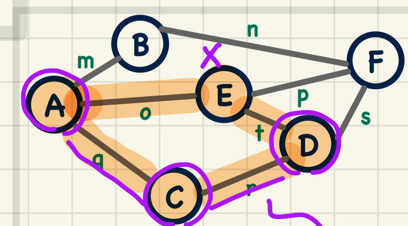
cycle

Simple Cycle: (E, t, $\boxed{D, r, C, q, A}$, o, E)

simple path (without any cycle)

Reachability Problem: Given G=(V, E); Is v1 reachable from v2?

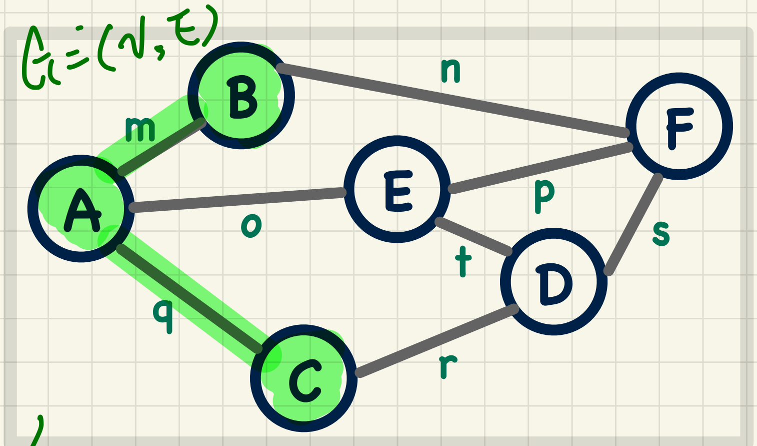
- $\boxed{\text{Path}}$ alternating v's and e's.
- Cycle
- $\boxed{\text{Simple Path}}$ → a path without cycle
- $\boxed{\text{Simple Cycle}}$ → 1. cycle 2. Except the two vertices forming the cycle, all remaining vertices are distinct
- $\boxed{\text{Reachable}}$
- Reachable Paths



a simple path

- subgraph
- spanning subgraph
- connected subgraph
- forest
- tree
- spanning tree

Graph: Subgraphs and Spanning Subgraphs



subgraph $G'=(V', E')$

$V' \subseteq V \wedge E' \subseteq E$

① G'_1 empty graph min subgraph

$V'=\emptyset \wedge E'=\emptyset$,

② G'_2 one-vertex graph $V'=\{A\} \wedge E'=\emptyset$ A

③ a subgraph cannot contain just a single edge.

④ $G'_3 = (\{A, B, C\}, \{m, q\})$

a subgraph but not spanning.

⑤ G can be its own subgraph!

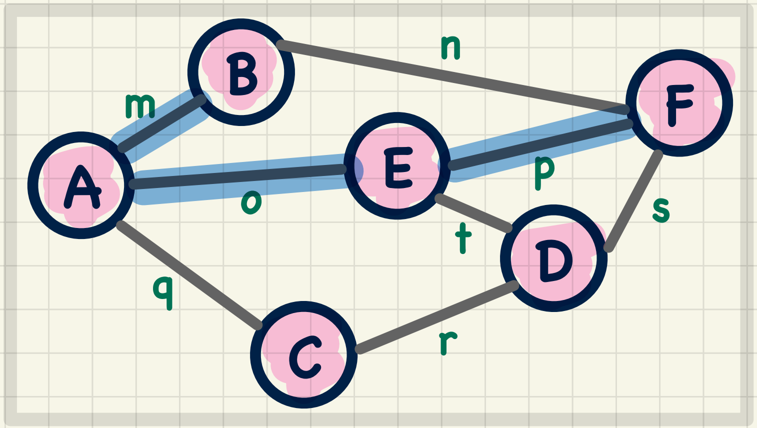
max subgraph

Spanning Subgraph

$\rightarrow$ a subgraph that "spans" through all vertices.

$\hookrightarrow$ $G' = (V', E')$ is a spanning subgraph of G

$\iff V' = V \wedge E' \subseteq E$



$\boxed{\text{spanning} \neq \text{connected}}$

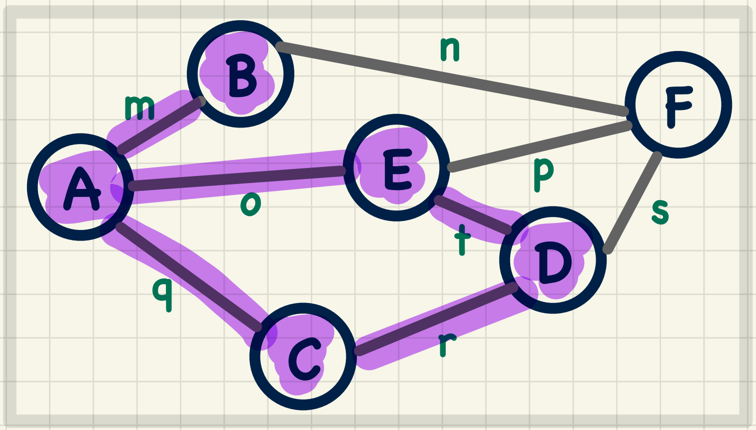
(1) $G_1' = (\underbrace{\{A, B, C, D, E, F\}}_{=V}, \underbrace{\emptyset}_{\subseteq E})$

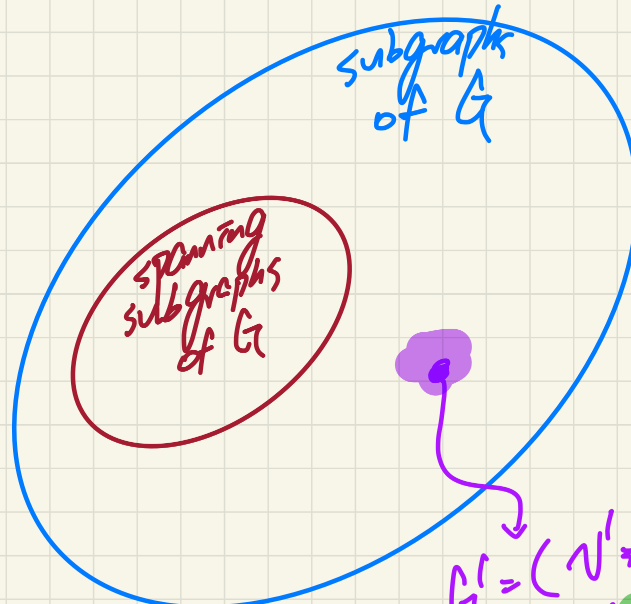
(2) $G_2' = (\underbrace{\{A, B, C, D, E, F\}}_{=V}, \underbrace{\{m, o, p\}}_{\subseteq E})$

Graph: Subgraphs and Spanning Subgraphs

$\subseteq V$ $\subseteq E$

Formulate a condition of a graph $G' = (V', E')$ that is a subgraph, but **not** a spanning subgraph, of $G = (V, E)$.





$G' = (V', E')$

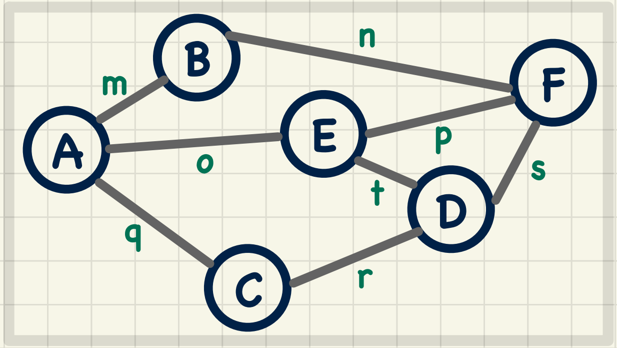
s.t. $V' \subset V \wedge E' \subseteq E$

Graph: Connected Graph

$G = (V, E)$

Connected$(G) \iff$

$\forall x, y \bullet x \in V \wedge y \in V \Rightarrow$ $\boxed{x \text{ is reachable}}$ from y.



only req. vertices to be covered, but not req. edges to build the necessary connections.

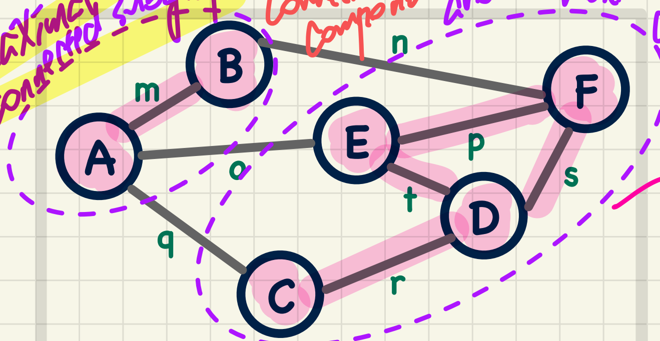
Is a spanning subgraph also connected subgraph?

Hint: Consider G2 = ({A, B, C, D, E, F}, {m, p, s, t, r})

maximal connected subgraph

Connected Component

another maximal connected subgraph



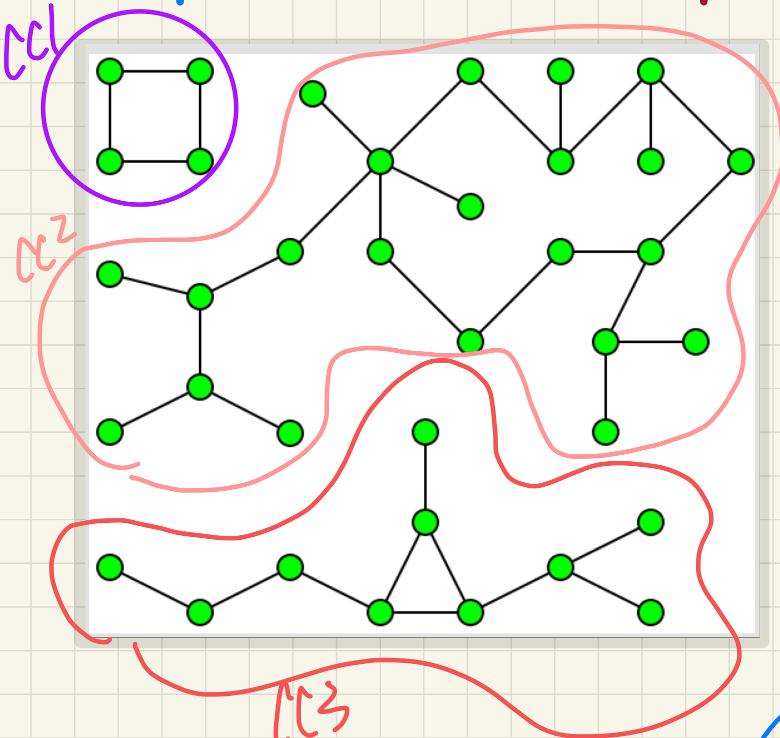
a spanning graph but not connected (e.g. no reachable path from B and F).

Connected Component of G

a $\boxed{\text{maximal}}$ connected subgraph of G

no further extension is possible to make a larger connected subgraph

Graph: Connected Components



How many connected components does the graph have?

Between each pair of CCs, say CC1 and CC2,

$\forall x, y \cdot x$ is a vertex in CC1 $\wedge$ y is a vertex in CC2

$\Rightarrow$ x is <u>not</u> reachable from y